Parametric Level Correlations in Random–Matrix Models

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Dedicated to Lothar Schäfer on the occasion of his sixtieth birthday

Abstract

We show that parametric level correlations in random–matrix theories are closely related to a breaking of the symmetry between the advanced and the retarded Green's functions. The form of the parametric level correlation function is the same as for the disordered case considered earlier by Simons and Altshuler and is given by the graded trace of the commutator of the saddle–point solution with the particular matrix that describes the symmetry breaking in the actual case of interest. The strength factor differs from the case of disorder. It is determined solely by the Goldstone mode. It is essentially given by the number of levels that are strongly mixed as the external parameter changes. The factor can easily be estimated in applications.

1 Introduction

Parametric level correlations in chaotic and disordered systems have received much attention in the early 1990's (see the review [1] and references therein). This development culminated in the seminal work of Simons and Altshuler [2] who showed that such correlations have a universal form and who calculated some of the correlation functions for disordered systems explicitly.

In the present paper, I take a fresh look at this problem. This is motivated by two circumstances. (i) The work of Simons and Altshuler does not address level correlations of random matrices (but rather of chaotic and/or disordered systems). But level correlations of random matrices do play a role in some applications of random–matrix theory. A case in point concerns correlations of levels with different spins in atomic and nuclear systems [3]. While the *form* of the correlation function obtained by Simons and Altshuler is expected to be unchanged, it is necessary to determine the dimensionless parameter which governs its behavior, and to connect that parameter with physical parameters of the system at hand. In particular, the concept of "level velocities" introduced by Simons and

Altshuler needs to be reconsidered. It will be shown that in contrast to the case of disorder, the strength parameter in random–matrix theory is not influenced by a coupling of the Goldstone mode with the massive modes. (ii) Parametric level correlations can be seen as a manifestation of symmetry breaking. The broken symmetry is that between the advanced and the retarded Green's functions. I aim at a presentation which displays this fact as clearly as possible. With this insight, writing down the form of the correlation funtion is quite straightforward.

The correlation functions will be given for the GOE and for the GUE. The results are also compared with the two–point correlation function for the GOE $\to$ GUE transition caused by time–reversal symmetry breaking. We shall see that in the latter case, symmetry breaking acts differently.

2 Formulation of the Problem

The ensemble of Hamiltonians H has the form

$$H = H_1 \cos(X) + H_2 \sin(X) \tag{1}$$

where X is a dimensionless parameter and where H_1 and H_2 are uncorrelated random matrices belonging to the same symmetry class of one of Dyson's three canonical ensembles. We wish to calculate the parametric correlation function

$$k = \overline{\mathrm{tr}[\frac{1}{E_1^+ - H(X)}]\mathrm{tr}[\frac{1}{E_2^- - H(X')}]} \ . \tag{2}$$

The overbar denotes the ensemble average. The function k contains quantitative information about the way in which the spectra at parameter values X and X' are correlated. For $X = X'$, k coincides with the standard two–point correlation function.

Both in the case of disordered systems and in the present case, one needs to calculate k only for small values of $|X - X'|$, i.e., perturbatively. The reason is that we are interested in local (rather than global) changes of the spectrum. The former involve an energy scale of order d, the mean level spacing, the latter, an energy scale of order Nd where $N \to \infty$ is the dimension of the matrices H_1 and H_2. Then, the function $k = k(\epsilon, X - X')$ depends only upon the difference $\epsilon = E_1 - E_2$ of the energies of the two Green's functions.

I expand the Hamiltonians $H(X)$ and $H(X')$ in Eq. (1) around the mid–point $X_0 = (1/2)(X + X')$ in powers of $X - X_0 = (1/2)(X - X')$ and of $X' - X_0 = (1/2)(X' - X)$, respectively, and keep only terms up to first order in $(X - X')$. Then,

$$\begin{aligned} H(X) &\approx H_0 + (1/2)(X - X')V \ , \\ H(X') &\approx H_0 - (1/2)(X - X')V \ , \end{aligned} \tag{3}$$

where $H_0 = H(X_0)$ and where

$$V = H_2 \cos(X_0) - H_1 \sin(X_0) \ . \tag{4}$$

The random matrices $H(X_0)$ and V are uncorrelated,

$$\overline{H_0 V} = 0 \ . \tag{5}$$

This follows from the fact that H_1 and H_2 are uncorrelated, $\overline{H_1 H_2} = 0$.

To identify the small parameter of the expansion, I define (as usual) the spreading width due to the perturbation as

$$\Gamma^{\downarrow} = 2\pi (X - X')^2 \overline{V^2}/d \ . \tag{6}$$

The spreading width is a measure of the energy interval within which the levels of H_0 get strongly mixed as the parameter changes from X to X'. We are interested in values of $\Gamma^{\downarrow}$ which are of the order of d (rather than Nd). We normalize the variances of H_1 and H_2 in the usual manner,

$$\overline{(H_j)_{\mu\nu}(H_j)_{\nu\mu}} = \frac{\lambda^2}{N} \ ; \ j = 1,2 \ ; \mu \neq \nu \ , \tag{7}$$

where μ and ν are level indices, and where 2λ is the radius of the semicircle. The mean level spacing of H_0 in the centre of the semicircle is given by $d = \pi\lambda/N$. With these conventions, we have

$$\Gamma^{\downarrow} = 2(X - X')^2 \lambda \ . \tag{8}$$

We shall see that the dimensionles parameter which governs the level correlation function is given by $\Gamma^{\downarrow}/d = (2/\pi)N(X-X')^2$. For this parameter to be of order unity, we must have that $(X - X')^2$ is of order $1/N$. This jusitifies our perturbation expansion and the fact that we keep only linear terms in $(X - X')$.

Substituting $H(X)$ and $H(X')$ from Eqs. (3) into Eq. (2) yields

$$\begin{aligned} & k(\epsilon, X - X') \\ &= \overline{\mathrm{tr}[\frac{1}{E_1^+ - H_0 - (1/2)(X - X')V}]\mathrm{tr}[\frac{1}{E_2^- - H_0 + (1/2)(X - X')V}]} \ . \end{aligned} \tag{9}$$

Eq. (9) displays explicitly the fact that the perturbation V breaks the symmetry between the retarded and the advanced Green's functions. This is essential for the supersymmetry calculation of $k(\epsilon, X - X')$.

3 Supersymmetry

The supersymmetry method [4, 5] has become a standard tool in random–matrix theory. Therefore, I confine myself to giving the essential steps in the calculation. I do so for the case where both H_1 and H_2 belong to the GOE, and give only results for the GUE.

I proceed as in Ref. [5], also use their notation, and arrive at the following form of the generating function,

$$Z(E_1, E_2; X, X', J) = \int \mathrm{d}[\Psi] \exp\{\mathcal{L}(\Psi, J)\} \ , \tag{10}$$

where the Lagrangian is given by

$$\mathcal{L} = (1/2) i (\Psi^\dagger L^{1/2} D^J L^{1/2} \Psi) \ . \tag{11}$$

Here D^J is a graded matrix of dimension 8, given by

$$D^J = (\mathbf{E} - \mathbf{H} + i\delta + \mathbf{J} - (1/2)\mathcal{E}) \ . \tag{12}$$

According to Eq. (3), the matrix $\mathbf{H}$ has the form

$$\mathbf{H} = H(X_0)\mathbf{1}_8 + (1/2)(X - X')VL \ . \tag{13}$$

Here $\mathbf{1}_8$ denotes the unit matrix in eight dimensions, while

$$L = \mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1) \tag{14}$$

is the matrix which breaks the symmetry between the advanced and the retarded Green's functions. We want to calculate the two–point function and accordingly put

$$\mathbf{J} = \delta_{\mu\nu}\mathrm{diag}(-j_1, -j_1, +j_1, +j_1, -j_2, -j_2, +j_2, +j_2) = \delta_{\mu\nu}(\mathbf{j}_1, \mathbf{j}_2) \ . \tag{15}$$

The last equation defines $(\mathbf{j}_1, \mathbf{j}_2)$.

The ensemble average is given in terms of the second moment of the term $(i/2)(\Psi^\dagger L^{1/2}\mathbf{H}L^{1/2}\Psi)$,

$$\begin{aligned}
&\overline{[(i/2)(\Psi^\dagger L^{1/2}\mathbf{H}L^{1/2}\Psi)]^2} \\
&= -(\lambda^2/(2N)) \sum_{\mu\nu\alpha\beta} \left(\Psi^\dagger_{\mu\alpha} (L^{1/2})_{\alpha\alpha} (L^{1/2})_{\alpha\alpha} \Psi_{\nu\alpha} \right) \\
&\qquad \times \left(\Psi^\dagger_{\nu\beta} (L^{1/2})_{\beta\beta} (L^{1/2})_{\beta\beta} \Psi_{\mu\beta} \right) \\
&\quad -(\lambda^2/(8N))(X - X')^2 \sum_{\mu\nu\alpha\beta} \left(\Psi^\dagger_{\mu\alpha} (L^{1/2})_{\alpha\alpha} L_{\alpha\alpha} (L^{1/2})_{\alpha\alpha} \Psi_{\nu\alpha} \right) \\
&\qquad \times \left(\Psi^\dagger_{\nu\beta} (L^{1/2}) \beta\beta L_{\beta\beta} (L^{1/2})_{\beta\beta} \Psi_{\mu\beta} \right) \ .
\end{aligned} \tag{16}$$

The summation over α, β runs from 1 to 8, that over μ, ν from 1 to N. I define

$$\begin{aligned}
A_{\alpha\beta} \quad = \quad & i\lambda \sum_\mu (L^{1/2})_{\alpha\alpha} \psi_{\mu\alpha} \psi^\dagger_{\mu\beta} (L^{1/2})_{\beta\beta} \\
& +(1/8)(X - X')^2 i\lambda \sum_\mu L_{\alpha\alpha} (L^{1/2})_{\alpha\alpha} \psi_{\mu\alpha} \psi^\dagger_{\mu\beta} (L^{1/2})_{\beta\beta} L_{\beta\beta} \ .
\end{aligned} \tag{17}$$

This equation clearly displays the separate contributions from H_0 and from the symmetry–breaking term VL. Under neglect of higher–order terms in $(X - X')^2$ (which we have shown to be negligible for $N \to \infty$), the right–hand side of Eq. (16) can be expressed in terms of A, yielding

$$\overline{[(i/2)(\Psi^\dagger L^{1/2}\mathbf{H}L^{1/2}\Psi)]^2} = \frac{1}{2N} \mathrm{trg}_\alpha (A^2) \ . \tag{18}$$

The Hubbard–Stratonovitch transformation yields now for Z the form

$$Z(E_1, E_2; X, X', J) = \int \mathrm{d}[\sigma] \exp\left\{ -\frac{N}{4}\mathrm{trg}_\alpha(\sigma^2) - \frac{N}{2}\mathrm{trg}_\alpha \ln \mathbf{N}(J) \right\} , \tag{19}$$

where

$$\mathbf{N}(J) = E\mathbf{1}_8 - (1/2)\mathcal{E} + i\delta - \lambda\Sigma + (\mathbf{j}_1, \mathbf{j}_2) \tag{20}$$

and

$$\Sigma = \sigma + (1/8)(X - X')^2 \ L\sigma L \ . \tag{21}$$

We use the saddle–point approximation, omitting terms which are of order $1/N$. These are the terms proportional to $\mathcal{E}$, to $(X' - X)^2$, and the source terms. The saddle–point equation

$$\sigma = \frac{\lambda}{E\mathbf{1}_8 - \lambda\sigma} \tag{22}$$

has the standard solution

$$\sigma_G = T_0^{-1}\sigma_D^0 T_0 \tag{23}$$

with σ_D^0 diagonal and given by

$$\sigma_D^0 = \frac{E}{2\lambda} - i\Delta_0 L \tag{24}$$

and $\Delta_0 = \sqrt{1 - (E/(2\lambda))^2}$. The full sigma matrix is written as

$$\sigma = \sigma_G + \delta\sigma = \sigma_G + T_0^{-1}\delta P T_0 \ . \tag{25}$$

It remains to work out the integrals over the massive modes, and over the Goldstone mode.

4 Integration over the Massive Modes

In applications of the supersymmetry formalism, one would normally skip the present Section because the integration over the massive modes is known to simply yield a constant. However, in the work of Simons and Altshuler [2], it is shown that the strength of the parametric level correlation function depends upon contributions due to the coupling of the Goldstone mode with the massive modes. Is such a mechanism also operative in the present case? To answer this question, I substitute in Eqs. (19) and (20) for Σ the expression (21) and in the latter for σ the expression (25). I expand in powers of $\delta\sigma$ and of the small entities $\mathcal{E}, (X' - X)^2$ and $(\mathbf{j}_1, \mathbf{j}_2)$ and keep terms up to the second order in $\delta\sigma$ and up to first order in the other small entities. Some of the linear terms in $\delta\sigma$ cancel because of the saddle–point condition. The exponent in Eq. (20) takes the form

$$-\frac{N}{4}\mathrm{trg}_\alpha\left[\delta\sigma\right]^2 + \frac{N}{4}\mathrm{trg}_\alpha\left[\sigma_G\delta\sigma\right]^2 + \frac{N\epsilon}{4\lambda}\mathrm{trg}_\alpha\left[\sigma_G L\right]$$

$$
\begin{aligned}
&-\frac{N}{2\lambda}\mathrm{trg}_\alpha\left[\sigma_G(\mathbf{j}_1,\mathbf{j}_2)\right]+\frac{N}{16}(X-X')^2\mathrm{trg}_\alpha\left[(\sigma_G L)^2\right]\\
&+\frac{N}{16}(X-X')^2\mathrm{trg}_\alpha\left[\sigma_G L\delta\sigma L\right]+\frac{N\epsilon}{4\lambda}\mathrm{trg}_\alpha\left[\sigma_G\delta\sigma\sigma_G L\right]\\
&-\frac{N}{2\lambda}\mathrm{trg}_\alpha\left[\sigma_G\delta\sigma\sigma_G(\mathbf{j}_1,\mathbf{j}_2)\right]\\
&+\frac{N}{16}(X-X')^2\mathrm{trg}_\alpha\left[\sigma_G\delta\sigma(\sigma_G L)^2+\sigma_G\delta\sigma\sigma_G L\delta\sigma L\right]\\
&+\frac{N\epsilon}{4\lambda}\mathrm{trg}_\alpha\left[\sigma_G L(\sigma_G\delta\sigma)^2\right]-\frac{N\epsilon}{2\lambda}\mathrm{trg}_\alpha\left[\sigma_G(\mathbf{j}_1,\mathbf{j}_2)(\sigma_G\delta\sigma)^2\right]\\
&+\frac{N}{16}(X-X')^2\mathrm{trg}_\alpha\left[(\sigma_G L)^2(\sigma_G\delta\sigma)^2\right]\ . \qquad (26)
\end{aligned}
$$

The leading terms in $(\delta\sigma)^2$ are the first two terms in expression (26). These terms show that all massive modes have mass N. We recall that $N(X-X')^2$ is of order unity. Therefore, the remaining terms which are quadratic in $\delta\sigma$ are negligible. The terms linear in $\delta\sigma$ are all at most of order unity. To be non–negligible, they ought to be of order $\sqrt{N}$. In the limit $N\to\infty$ we are, thus, left with the first five terms in expression (26). This shows that the massive modes decouple from the Goldstone mode. Moreover, the massive–mode contribution attains exactly the form given in Ref. [5] and can, therefore, be integrated out without any problem. Hence, in contrast to the disorder problem studied in Ref. [2], the massive modes do not contribute to the strength of the parametric level correlation function in random–matrix theory.

The result for Z is

$$
\begin{aligned}
Z(E_1,E_2;X,X',J) \;=\; & 4\int \mathrm{d}[\sigma]\exp\Bigg\{+\frac{\pi\epsilon}{4d}\mathrm{trg}_\alpha(\sigma_G L)\\
&+\frac{N}{16}(X-X')^2\mathrm{trg}_\alpha(\sigma_G L)^2\Bigg\}\\
&\times\frac{N^2}{8\lambda^2}\Big(\mathrm{trg}_\alpha[(\mathbf{j}_1,\mathbf{j}_2)\sigma_G]\Big)^2\ . \qquad (27)
\end{aligned}
$$

I carry out the differentiation with respect to j_1 and j_2. The result is

$$
\begin{aligned}
k(\epsilon,X-X') \;=\; & (1/2)\int \mathrm{d}[\sigma]\exp\Bigg\{+\frac{\pi\epsilon}{4d}\mathrm{trg}_\alpha(\sigma_G L)\\
&+\frac{N}{16}(X-X')^2\mathrm{trg}_\alpha(\sigma_G L)^2\Bigg\}\\
&\times\frac{\pi^2}{d^2}(\mathrm{trg}_\alpha[I(1)(\sigma_G)_{1,1}])(\mathrm{trg}_\alpha[I(2)(\sigma_G)_{2,2}])\ . \qquad (28)
\end{aligned}
$$

5 Integration over the Goldstone Mode

Our result Eq. (28) differs from the standard expression for the GOE two–point function by an additional term appearing in the exponent. Using

Eq. (8), we rewrite this term in the form

$$+\frac{N}{16}(X-X')^2\mathrm{trg}_\alpha(\sigma_G L)^2 = +\frac{\pi\Gamma^\downarrow}{64d}\mathrm{trg}_\alpha[([\sigma_G,L])^2]\ . \tag{29}$$

Once again, the right–hand side of this equation shows very clearly that the term is due to the symmetry breaking caused by the perturbation.

In the three graded traces appearing on the right–hand side of Eq. (28), the only matrices which break the pseudounitary symmetry are $I(1)$ and $I(2)$. Therefore, the two graded traces in the exponent depend only upon the "eigenvalues" (remaining integration variables). I use the parametrizations of both Refs. [5] and [4] to work out Z in the middle of the spectrum where $\Delta_0 = 1$.

For the parametrization of Ref. [5] I find

$$\begin{aligned} k(\epsilon, X-X') \quad \propto \quad & \int_0^\infty \mathrm{d}\lambda_1 \int_0^\infty \mathrm{d}\lambda_2 \int_0^1 \mathrm{d}\lambda \\ & \times \frac{(1-\lambda)\lambda|\lambda_1-\lambda_2|}{((1+\lambda_1)\lambda_1(1+\lambda_2)\lambda_2)^{1/2}(\lambda+\lambda_1)^2(\lambda+\lambda_2)^2} \\ & \times \exp\Bigg\{-\frac{i\pi\epsilon}{d}(\lambda_1+\lambda_2+2\lambda) \\ & \quad -\frac{\pi\Gamma^\downarrow}{4d}(\lambda_1+\lambda_2+2\lambda)(1+\lambda_1+\lambda_2+2\lambda)\Bigg\} \\ & \times(\lambda_1+\lambda_2+2\lambda)^2\ . \end{aligned} \tag{30}$$

For the parametrization of Ref. [4], the two terms in the exponent take the form

$$\frac{i\pi\omega}{d}(\lambda-\lambda_1\lambda_2) - \frac{\pi\Gamma^\downarrow}{4d}(2\lambda_1^2\lambda_2^2-\lambda_1^2-\lambda_2^2-\lambda^2+1)\ . \tag{31}$$

The first term agrees with Efetov's Eq. (5.35) if the definition of x following this equation is taken into account. The combination of integration variables appearing in the second term is the same as given by Simons and Altshuler.

Hopefully, the derivation given above shows very clearly the role of symmetry breaking in parametric level correlations. The result confirms our expectation: The form of the parametric level correlation function is the same as for the disordered case. The strength factor differs and is given by $\pi\Gamma^\downarrow/(4d)$. Except for the numerical factor $\pi/4$, this result, too, corresponds to naive expectations: $\Gamma^\downarrow/d$ is a measure of the number of levels which are strongly mixed with each other as the external parameter changes from X to X'. In applications, this paramter can be estimated in terms of the strength of the perturbation and of the local mean level spacing.

6 General Aspects of Symmetry Breaking

I now address more fully the symmetry–breaking mechanism which occurs when one considers parametric level correlations between two Hamiltonian

ensembles H_1 and H_2 (symbolically denoted by $H_1 \longleftrightarrow H_2$). I do so in several situations with the intention of exhibiting the underlying similarities and differences. I consider the following cases: (i) GOE $\longleftrightarrow$ GOE; (ii) GUE $\longleftrightarrow$ GUE; (iii) GOE $\longleftrightarrow$ GUE. In the last case, the two Hamiltonians H_1 and H_2 obviously do not belong to the same symmetry class. For the sake of comparison, I consider also (iv) the two–point autocorrelation function for the GOE $\rightarrow$ GUE transition [6]. Proceeding as before, I calculate the resulting contributions to the effective Lagrangean (i.e., the additional symmetry–breaking terms in the exponent). These are jointly denoted by S and referred to as the parametric correlator. The commuting (c) and anticommuting (a) integration variables are arranged as follows: For the GUE two–point function, the sequence is (c, a, c, a) while for the GOE, it is (c, c, a, a, c, c, a, a). I define the graded matrix

$$\mathrm{T}_3 = \mathrm{diag}(+1, -1, +1, -1; +1, -1, +1, -1) \ . \tag{32}$$

Case (i) has been considered above. The parametric correlator was found to have the form $S_{(\mathrm{i})} = \mathrm{trg}\{([\sigma_G, L])^2\}$. Case (ii) is formally very similar and leads to the same expression except that now σ_G and L have dimension four rather than eight. In case (iii), we get the GUE by adding to the GOE matrix in the advanced Green's function an imaginary random matrix. Thus, the new term in the exponent arises only from the advanced Green's function (and not from both, the retarded and the advanced Green's function as in the previous cases (i) and (ii)). Moreover, the new term carries the matrix τ_3 as the signal for the breaking of GOE symmetry and suppression of the Cooperon mode. As a result, the relevant term has the form $S_{(\mathrm{iii})} = \mathrm{trg}\{([\sigma_G, (\mathbf{1}_8 - L_8)\mathrm{T}_3])^2\}$. Case (iv) leads to a symmetry–breaking term of the form $S_{(\mathrm{iv})} = \mathrm{trg}\{([\sigma_G, \mathrm{T}_3])^2\}$. This obviously differs from $S_{(\mathrm{iii})}$. I observe that all these paramertic correlators have the same form, $\mathrm{trg}\{([\sigma_G, T_\mathrm{x}])^2\}$, with T_x given by

$$\begin{aligned} T_\mathrm{i} &= L_8 \ , \\ T_\mathrm{ii} &= L_4 \ , \\ T_\mathrm{iii} &= (1_8 - L_8)\mathrm{T}_3 \ , \\ T_\mathrm{iv} &= \mathrm{T}_3 \ . \end{aligned} \tag{33}$$

In summary, we have shown that the parametric correlation functions in random–matrix theory have a very simple form. Each one is obtained from the standard two–point function for level correlations by adding in the exponent of the generating function a term. That term is given by the graded trace of the commutator of the saddle–point solution σ_G with the particular matrix that describes the symmetry breaking in the actual case of interest. Except for a numerical factor which is of order unity, the factor in front of the commutator is given by $\Gamma^{\downarrow}/d$.

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